JOURNAL OF APPROXIMATION THEORY 69, 302–317 (1992)

On the L^p Convergence of Lagrange Interpolating Entire Functions of Exponential Type

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Communicated by D. S. Lubinsky

Received October 24, 1990; revised April 17, 1991

Let $f: \mathbb{R} \mapsto \mathbb{C}$ be a continuous, 2π-periodic function and for each $n \in \mathbb{N}$ let $t_n(f; \cdot)$ denote the trigonometric polynomial of degree $\leqslant n$ interpolating f in the points $2k\pi/(2n+1)$ $(k=0, \pm 1, ..., \pm n)$. It was shown by J. Marcinkiewicz that $\lim_{n \to \infty} \int_0^{2\pi} |f(\theta) - t_n(f; \theta)|^p \, d\theta = 0$ for every $p > 0$. We consider Lagrange interpolation of non-periodic functions by entire functions of exponential type $\tau > 0$ in the points $k\pi/\tau$ $(k = 0, \pm 1, \pm 2, ...)$ and obtain a result analogous to that of Marcinkiewicz. © 1992 Academic Press, Inc.

1. INTRODUCTION

For each $n \in \mathbb{N}$ let

$$\theta_{n,k} := \frac{2k\pi}{2n+1} \qquad (k = 0, \pm 1, ..., \pm n)$$

and denote by $t_n(f; \cdot)$ the trigonometric interpolatory polynomial of degree not exceeding n with $t_n(f; \theta_{n,k}) = f(\theta_{n,k})$. It was shown by Marcinkiewicz [10] that if $f: \mathbb{R} \to \mathbb{C}$ is a continuous, 2π-periodic function, then for every $p > 0$

$$\lim_{n \to \infty} \int_0^{2\pi} |f(\theta) - t_n(f; \theta)|^p \, d\theta = 0. \tag{1}$$

* This author was partially supported by the Hungarian National Foundation for Scientific Research Grant 1801.

0021-9045/92 $5.00

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The interest of this result lies in the fact that $\limsup_{n\to\infty} |t_n(f;\theta)| = \infty$ for every θ if the continuous and 2π-periodic function f is suitably chosen (see [7, 11]).

We consider Lagrange interpolation of non-periodic functions by entire functions of exponential type $\tau > 0$ in the points

$$x_{\tau,k} := \frac{k\pi}{\tau} \qquad (k = 0, \pm 1, \pm 2, ...)$$

and obtain a result analogous to that of Marcinkiewicz. In order to place our result in perspective we recall that every continuous function $f\colon \mathbb{R} \to \mathbb{C}$ can be approximated arbitrarily closely on $\mathbb{R}$ by entire functions [4]. It was shown by Bernstein [1] that if f is continuous and bounded on $\mathbb{R}$ and E^τ is the class of all entire functions of exponential type τ bounded on $\mathbb{R}$ then

$$A_\tau(f) := \inf_{F\in E^\tau} \sup_{x\in\mathbb{R}} |f(x) - F(x)|$$

tends to zero as $\tau \to \infty$ if and only if f is uniformly continuous. To f we associate

$$L_\tau(f; z) := \sum_{k=-\infty}^{\infty} f(x_{\tau,k})\, g_{\tau,k}(z) \tag{2}$$

where for $k = 0, \pm 1, \pm 2, ...$

$$g_{\tau,k}(z) := \begin{cases} \dfrac{\sin \tau(z - x_{\tau,k})}{\tau(z - x_{\tau,k})} & \text{if} \quad z \neq x_{\tau,k} \\ 1 & \text{if} \quad z = x_{\tau,k}, \end{cases} \tag{3}$$

and investigate if it converges to f (in one norm or the other) as $\tau \to \infty$. We are able to show that for every $p > 1$

$$\|f - L_\tau(f;\cdot)\|_p := \left(\int_{-\infty}^{\infty} |f(x) - L_\tau(f;x)|^p\, dx\right)^{1/p} \to 0 \qquad \text{as} \quad \tau \to \infty \tag{4}$$

if for some $\delta > 0$

$$f(x) = O\left(\frac{1}{(|x|+1)^{1/p+\delta}}\right) \qquad (x \in \mathbb{R}). \tag{5}$$

We wish to point out that $\sup_{x\in\mathbb{R}} |f(x) - L_\tau(f;x)|$ may not tend to zero as $\tau \to \infty$, if f satisfies (5). Indeed, if X denotes the Banach space of all continuous functions $f: \mathbb{R} \to \mathbb{C}$ which vanish outside $[0, 1]$ then $f \to L_\tau(f; \cdot)$ defines a bounded linear transformation Λ_τ from X to the normed linear space Y of all continuous functions φ on $[-1, 1]$ with $\|\varphi\| := \max_{-1 \leqslant x \leqslant 1} |\varphi(x)|$. Using $[\tau/\pi]$ to denote the integral part of τ/π we see that for all large τ

$$\|\Lambda_\tau\| \geqslant \sum_{k=1}^{[\tau/\pi]} \left| g_{\tau,k}\left(\frac{\pi}{2\tau}\right)\right| = \frac{2}{\pi} \sum_{k=1}^{[\tau/\pi]} \frac{1}{2k-1} > \frac{1}{\pi} \log \tau - 1,$$

i.e., $\sup_\tau \|\Lambda_\tau\| = \infty$. Hence by the Banach–Steinhaus theorem [13, p. 98] there exists a function $f^* \in X$ and so one satisfying (5) such that

$$\max_{-1 \leqslant x \leqslant 1} |f^*(x) - L_\tau(f^*; x)|$$

does not remain bounded as $\tau \to \infty$. This idea is essentially contained in [6, pp. 211–212]; there is a slight difference nevertheless.

Definition 1. Given $p > 1$, we denote by $\mathfrak{F}^p(\delta)$ the set of all measurable functions $f: \mathbb{R} \to \mathbb{C}$ satisfying (5) for some $\delta > 0$ and by $\mathfrak{F}^p$ the union $\bigcup_{\delta > 0} \mathfrak{F}^p(\delta)$. Clearly $\mathfrak{F}^p \subset L^p(\mathbb{R})$.

If $f \in \mathfrak{F}^p(\delta)$ then $f \in \mathfrak{F}^p(\gamma)$ for every $\gamma < \delta$. So we may and indeed will assume $0 < \delta < 1 - 1/p$. It is clear that if $f \in \mathfrak{F}^p$, then

$$\sum_{k=-\infty}^{\infty} \left| f\left(\frac{k\pi}{\tau}\right)\right|^p < \infty. \tag{6}$$

Definition 2. We denote by $\mathfrak{R}$ the set of all functions $f: \mathbb{R} \to \mathbb{C}$ which are Riemann integrable on every finite interval.

With this we are ready to state our analogue of Marcinkiewicz's result mentioned above.

Theorem 1. *Let $p > 1$. Then (4) holds if $f \in \mathfrak{F}^p \cap \mathfrak{R}$.*

Theorem 1 is related to the well-known sampling theorem which plays an important role in communication, control theory, and data processing. In the language of electrical engineers the difference $f - L_\tau(f; \cdot)$ is called the *aliasing error* and a function $f \in C(\mathbb{R})$ with compact support is referred to as a *duration limited signal* (for these and other terms used by them in this connection see [3] and some of the papers quoted therein). Since a duration limited signal f trivially satisfies condition (5), Theorem 1 applies and so the following corollary holds.

COROLLARY 1. *For a duration limited Riemann integrable (finite energy) signal the L^2 norm of aliasing error can be made arbitrarily small.*

For a uniform bound of the aliasing error additional assumptions are needed and are usually stated as conditions on the spectrum.

2. AUXILIARY RESULTS AND PREPARATORY LEMMAS

The entire functions $g_{\tau,k}$ introduced in (3) are of exponential type τ and belong to $L^p(\mathbb{R})$ for every $p>1$. In particular, they belong to the class E^τ and further

$$g_{\tau,j}(x_{\tau,k})=\delta_{j,k}.$$

Now we collect some known facts from the theory of entire functions of exponential type and prove some preliminary results.

LEMMA 1 [12 or 2, Theorem 6.7.1]. *If g is an enitre function of exponential type τ and $\int_{-\infty}^{\infty}|g(x)|^p\,dx<\infty$, $p>0$, then*

$$\left(\int_{-\infty}^{\infty}|g(x+iy)|^p\,dx\right)^{1/p}\leqslant e^{\tau|y|}\left(\int_{-\infty}^{\infty}|g(x)|^p\,dx\right)^{1/p}. \tag{7}$$

Moreover, $g(x)\to 0$ as $x\to\pm\infty$.

This is analogous to the well-known fact that if $f\in E^\tau$, then

$$\sup_{-\infty<x<\infty}|f(x+iy)|\leqslant e^{\tau|y|}\sup_{-\infty<x<\infty}|f(x)|.$$

LEMMA 2 [12 or 2, Theorem 6.7.15]. *Under the conditions of Lemma 1*

$$\left(\frac{\pi}{\tau}\sum_{k=-\infty}^{\infty}\left|g\left(\frac{k\pi}{\tau}\right)\right|^p\right)^{1/p}\leqslant A_p\left(\int_{-\infty}^{\infty}|g(x)|^p\,dx\right)^{1/p} \tag{8}$$

where A_p depends on p only.

The next lemma contains some useful information about the function $L_\tau(f;\cdot)$ associated with an $f\in\mathfrak{F}^p$.

LEMMA 3. *Let $p>1$. If $f\in\mathfrak{F}^p(\delta)$ then* (i) $L_\tau(f;\cdot)\in E^\tau$ *and* (ii) $L_\tau(f;\cdot)\in\mathfrak{F}^p(\gamma)$ *for each $\gamma\in(0,\delta)$.*

Proof. (i) For $z\in\mathbb{C}$, $\zeta\in\mathbb{C}$ let

$$h_\tau(z,\zeta):=\frac{\sin\tau(z-\zeta)}{\tau(z-\zeta)}.$$

If $z=x+iy$ is fixed, then as a function of ζ, $h_\tau(z,\zeta)$ is entire and of exponential type τ belonging to $L^q(\mathbb{R})$ for all $q>1$. Hence if $q>1$, then by Lemma 1

$$\int_{-\infty}^{\infty} |h_\tau(z,\xi)|^q\, d\xi = \int_{-\infty}^{\infty} \left|\frac{\sin \tau(\xi+iy)}{\tau(\xi+iy)}\right|^q d\xi \leqslant \frac{1}{\tau} e^{q\tau|y|} \int_{-\infty}^{\infty} \left|\frac{\sin \xi}{\xi}\right|^q d\xi$$

which in conjunction with Lemma 2 gives

$$\left(\sum_{k=-\infty}^{\infty} \left|h_\tau\left(z, \frac{k\pi}{\tau}\right)\right|^q\right)^{1/q} \leqslant A_q \left(\frac{1}{\pi}\int_{-\infty}^{\infty} \left|\frac{\sin \xi}{\xi}\right|^q d\xi\right)^{1/q} e^{\tau|y|} \tag{9}$$

This inequality allows us to conclude that if $q=p/(p-1)$ then for N_1, $N_2 \in \mathbb{Z}$, $N_1<N_2$,

$$\begin{aligned}
&\left|\sum_{k=N_1}^{N_2} f\left(\frac{k\pi}{\tau}\right) h_\tau\left(z, \frac{k\pi}{\tau}\right)\right| \\
&\quad\leqslant \left(\sum_{k=N_1}^{N_2} \left|f\left(\frac{k\pi}{\tau}\right)\right|^p\right)^{1/p} \left(\sum_{k=-\infty}^{\infty} \left|h_\tau\left(z, \frac{k\pi}{\tau}\right)\right|^q\right)^{1/q} \\
&\quad\leqslant \left(\sum_{k=N_1}^{N_2} \left|f\left(\frac{k\pi}{\tau}\right)\right|^p\right)^{1/p} A_q \left(\frac{1}{\pi}\int_{-\infty}^{\infty} \left|\frac{\sin \xi}{\xi}\right|^q d\xi\right)^{1/q} e^{\tau|y|}.
\end{aligned}$$

Hence, in view of (6), the series $\sum_{k=-\infty}^{\infty} f(k\pi/\tau) h_\tau(z, k\pi/\tau)$ converges uniformly on all compact subsets of $\mathbb{C}$ and so its sum, which is $L_\tau(f;z)$ (because $h_\tau(z, k\pi/\tau) = g_{\tau,k}(z)$), defines an entire function. Further

$$|L_\tau(f;z)| \leqslant \left(\sum_{k=-\infty}^{\infty} \left|f\left(\frac{k\pi}{\tau}\right)\right|^p\right)^{1/p} A_q \left(\frac{1}{\pi}\int_{-\infty}^{\infty} \left|\frac{\sin \xi}{\xi}\right|^q d\xi\right)^{1/q} e^{\tau|y|} \tag{10}$$

which implies that $L_\tau(f;\cdot)$ is of exponential type τ and is bounded on the real axis.

(ii) Let $x \in [j\pi/\tau, (j+1)\pi/\tau)$, where $j \in \mathbb{Z}$. Since $|g_{\tau,k}(x)| \leqslant 1$ for $x \in \mathbb{R}$ we readily obtain

$$|g_{\tau,k}(x)| \leqslant \frac{2}{|j-k|+1} \qquad \text{for} \quad k=j, j+1.$$

Now note that $|x-x_{\tau,k}|$ is bounded below by $(j-k)\pi/\tau$ if $k \leqslant j-1$ and by $(k-j-1)\pi/\tau$ if $k \geqslant j+2$. As such, $|g_{\tau,k}(x)| \leqslant 2/(|j-k|+1)$ also for $k \neq j$, $j+1$, i.e.,

$$|g_{\tau,k}(x)| \leqslant \frac{2}{|j-k|+1} \qquad \text{for all } k. \tag{11}$$

By assumption there exists a constant C_1 such that $|f(x)| < C_1/(|x|+1)^{1/p+\delta}$ for all $x \in \mathbb{R}$. Hence for large positive x

$$|L_\tau(f;x)| < 2C_1 \left\{\frac{1}{j+1} + 2\left(\frac{\tau}{\pi}\right)^{1/p+\delta} \right.$$
$$\left. \times \left(\sum_{k=1}^{j} \frac{1}{(j-k+1)k^{1/p+\delta}} + \sum_{k=j+1}^{\infty} \frac{1}{(k-j+1)k^{1/p+\delta}} \right) \right\}.$$

In order to estimate the two sums on the right we break them into two parts each, thus obtaining four sums. In the first k varies from 1 to $[j/2]$, in the second from $[j/2]+1$ to j, in the third from $j+1$ to $2j-1$, and in the fourth from $2j$ to ∞. We then readily see that for some constant C_2

$$|L_\tau(f;x)| < C_2 \left(\frac{1}{j} + \frac{1}{j^{1/p+\delta}} + \frac{1}{j^{1/p+\delta}} \log j \right) < C_2 \frac{3}{j^{1/p+\delta}} \log j.$$

Hence the desired result holds for positive x. But then it must also hold for negative x.

Lemma 3 helps us to prove in particular

LEMMA 4. *The transformation $f \to L_\tau(f;\cdot)$ reproduces entire functions of exponential type τ belonging to $\mathfrak{F}^p$.*

Proof. If $\varphi(z) := L_\tau(f;z) - f(z)$ then $\psi(z) := \varphi(\pi z/\tau)$ is an entire function which, in view of (10), satisfies

$$|\psi(z)| = O(e^{\pi|z|}), \qquad z \in \mathbb{C}.$$

Since $\psi(z) = 0$ for $z = 0, \pm 1, \pm 2, \ldots$ we can use a result of Pólya [2, Corollary 9.4.2] to conclude that $\psi(z) \equiv c \sin \pi z$. But $\psi(x) \to 0$ as $x \to \pm\infty$ since f and $L_\tau(f;\cdot)$ belong to $\mathfrak{F}^p$; as such, the constant c must be zero. Hence $\psi(z) \equiv 0$ which implies $L_\tau(f;z) \equiv f(z)$.

Given $f\colon \mathbb{R} \to \mathbb{C}$ and $\tau > 0$, consider the integral

$$S_\tau(f;z) := \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \frac{\sin \tau(z-t)}{z-t}\, dt$$

which certainly exists if $f \in L^p(\mathbb{R})$ for some $p > 1$. For *real* z we may also write

$$S_\tau(f;z) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(z+t) \frac{\sin \tau t}{t}\, dt.$$

The function $S_\tau(f;\cdot)$ has many interesting properties. For example, if $f\in L^p(\mathbb{R})$ for some $p>1$ then $S_\tau(f;z)$ is an entire function of exponential type τ and $\|S_\tau(f;\cdot)-f\|_p\to 0$ as $\tau\to\infty$. Lemmas 5–9 contain facts about $S_\tau(f;\cdot)$ which we need for the proof of our theorem on the L^p convergence of $L_\tau(f;\cdot)$.

From Hölder's inequality follows

LEMMA 5. *If $f\in L^p(\mathbb{R})$ for some $p>1$, then*

$$\|S_\tau(f;\cdot)\|_p\leqslant K_p\,\|f\|_p \tag{12}$$

where K_p depends on p only.

Next we prove

LEMMA 6. *If $f\in L^p(\mathbb{R})$ for some $p>1$, then $S_\tau(f;\cdot)\in E^\tau$.*

Proof. The entire function $g(z):=(\sin\tau(z-t))/(z-t)$ belongs to $L^q(\mathbb{R})$ for every $q>1$ and so for $q=p/(p-1)$. Since it is also of exponential type τ, Lemma 1 implies

$$\left(\int_{-\infty}^{\infty}|g(x+iy)|^q\,dx\right)^{1/q}\leqslant e^{\tau|y|}\left(\int_{-\infty}^{\infty}|g(x)|^q\,dx\right)^{1/q}$$

and therefore

$$\begin{aligned}|S_\tau(f;x+iy)|&\leqslant\frac{1}{\pi}\left(\int_{-\infty}^{\infty}|f(t)|^p\,dt\right)^{1/p}\left(\int_{-\infty}^{\infty}|g(x)|^q\,dx\right)^{1/q}e^{\tau|y|}\\&=O\,(e^{\tau|y|}.\end{aligned}$$

This shows that $S_\tau(f;\cdot)$ is of exponential type τ; that it is entire is obvious.

LEMMA 7. *If $f\in\mathfrak{F}^p$ then $S_\tau(f;\cdot)\in\mathfrak{F}^p$.*

Proof. Throughout this proof C will denote a positive constant not necessarily the same at each occurrance. We assume, as we may, that $f\in\mathfrak{F}^p(2\varepsilon)$, where $\varepsilon<(p-1)/2p^2$. Let x be large and positive. In order to estimate $S_\tau(f;x)$ we express it as the sum of three integrals,

$$S_\tau(f;x)=\left(\int_{-\infty}^{-2x}+\int_{-2x}^{-x/2}+\int_{-x/2}^{\infty}\right)\frac{1}{\pi}f(x+t)\frac{\sin\tau t}{t}\,dt=:I_1+I_2+I_3.$$

If $t\in(-\infty,-2x]$ then $t<x+t<t/2$ and so $|f(x+t)|<C/(|t|+2)^{1/p+2\varepsilon}$. This gives

$$|I_1|<\int_{2x}^{\infty}\frac{C}{t^{1/p+2\varepsilon+1}}\,dt<\frac{C}{x^{1/p+2\varepsilon}}. \tag{13}$$

The assumption on ε implies that $p_1 := p - p^2\varepsilon > 1$. So if $q_1 := p_1/(p_1 - 1)$ then by Hölder's inequality

$$|I_2| \leqslant \frac{1}{\pi}\left(\int_{-2x}^{-x/2} |f(x+t)|^{p_1}\, dt\right)^{1/p_1} \left(\int_{-2x}^{-x/2} \left|\frac{\sin \tau t}{t}\right|^{q_1} dt\right)^{1/q_1}$$

$$< C\left(\int_{x/2}^{2x} \frac{1}{(|x-u|+1)^{(1/p+2\varepsilon)p_1}}\, du\right)^{1/p_1} \left(\int_{x/2}^{2x} t^{-q_1}\, dt\right)^{1/q_1}.$$

Again by our assumption on ε

$$\left(\frac{1}{p} + 2\varepsilon\right) p_1 = 1 + p\varepsilon - 2p^2\varepsilon^2 > 1 + \varepsilon$$

and so

$$|I_2| < \frac{C}{x^{1/p_1}} < \frac{C}{x^{1/p+\varepsilon}}. \tag{14}$$

Finally, with $x + t = u$,

$$|I_3| < C\int_{x/2}^{\infty} \frac{1}{(u+1)^{1/p+\varepsilon}} \frac{|\sin \tau(u-x)|}{(u+1)^{\varepsilon}\,|u-x|}\, du$$

$$< \frac{C}{(x+2)^{1/p+\varepsilon}} \int_{x/2}^{\infty} \frac{|\sin \tau(u-x)|}{|u-x|^{1+\varepsilon}}\, du$$

$$= \frac{C}{(x+2)^{1/p+\varepsilon}} \left(\int_{x/2}^{x-1} + \int_{x-1}^{x} + \int_{x}^{x+1} + \int_{x+1}^{\infty}\right) \frac{|\sin \tau(u-x)|}{|u-x|^{1+\varepsilon}}\, du$$

$$< \frac{C}{x^{1/p+\varepsilon}}. \tag{15}$$

From (13)–(15) it follows that $|S_\tau(f; x)| < C/|x|^{1/p+\varepsilon}$ for large and positive x. Due to obvious summetry, the same estimate must also hold as $x \to -\infty$. In particular $S_\tau(f; \cdot) \in \mathfrak{F}^p$.

Lemma 4, in conjunction with Lemmas 6 and 7, gives

LEMMA 8. *If $f \in \mathfrak{F}^p$ then $L_\tau(S_\tau(f; \cdot)) = S_\tau(f; \cdot)$.*

We also need the following

LEMMA 9. *Let $f \in L^p(\mathbb{R})$ for some $p > 1$. Then*

$$\lim_{\tau \to \infty} \int_{-\infty}^{\infty} |S_\tau(f; x) - f(x)|^p\, dx = 0. \tag{16}$$

Proof. We start by proving (16) for the characteristic function χ of the interval [0, 1]. For this we need the easily verifiable facts that

$$\left|\int_0^T \frac{\sin t}{t}\,dt\right| \leqslant \frac{\pi}{2}+\frac{2}{\pi} \tag{17}$$

for all T and that

$$\lim_{T\to\infty}\int_0^T \frac{\sin t}{t}\,dt=\frac{\pi}{2}. \tag{18}$$

Simple calculation gives

$$S_\tau(\chi; x)=\frac{1}{\pi}\int_0^{\tau x}\frac{\sin t}{t}\,dt+\frac{1}{\pi}\int_0^{\tau(1-x)}\frac{\sin t}{t}\,dt.$$

From (18) it follows that for every $\delta>0$ there exists $T_0(\delta)$ such that

$$\left|\frac{1}{\pi}\int_0^T\frac{\sin t}{t}\,dt-\frac{1}{2}\right|<\frac{\delta}{2}$$

if $T>T_0(\delta)$. If η is a *fixed* number in $(0, \frac{1}{2})$ then for $\tau>(1/\eta)T_0(\delta)$ both τx and $\tau(1-x)$ are larger than $T_0(\delta)$ and so

$$|S_\tau(\chi; x)-1|\leqslant\left|\frac{1}{\pi}\int_0^{\tau x}\frac{\sin t}{t}\,dt-\frac{1}{2}\right|+\left|\frac{1}{\pi}\int_0^{\tau(1-x)}\frac{\sin t}{t}\,dt-\frac{1}{2}\right|<\delta$$

if $x\in[\eta, 1-\eta]$. Similarly, if $x\geqslant 1+\eta$ or if $x\leqslant-\eta$ then for $\tau>(1/\eta)T_0(\delta)$

$$|S_\tau(\chi; x)|\leqslant\left|\frac{1}{\pi}\int_0^{\tau|x|}\frac{\sin t}{t}\,dt-\frac{1}{2}\right|+\left|\frac{1}{\pi}\int_0^{\tau(1-x)}\frac{\sin t}{t}\,dt-\frac{1}{2}\right|<\delta.$$

Thus if $E_\eta := \{x : |x|<\eta \text{ or } |x-1|<\eta\}$ then

$$|S_\tau(\chi; x)-\chi(x)|<\delta \qquad \text{for all} \quad x\in\mathbb{R}\setminus E_\eta \tag{19}$$

if $\tau>(1/\eta)T_0(\delta)$.

For $x\geqslant A>1$

$$|S_\tau(\chi; x)-\chi(x)|=|S_\tau(\chi; x)|=\left|\frac{1}{\pi}\int_{\tau(x-1)}^{\tau x}\frac{\sin t}{t}\,dt\right|<\frac{1}{\pi(x-1)}$$

and so for given $\varepsilon>0$

$$\int_A^\infty |S_\tau(\chi; x)-\chi(x)|^p\,dx<\frac{\varepsilon}{4} \qquad \text{if} \quad A\geqslant A_0(\varepsilon). \tag{20}$$

Similarly

$$\int_{-\infty}^{-A} |S_\tau(\chi; x) - \chi(x)|^p \, dx < \frac{\varepsilon}{4} \qquad \text{if} \quad A \geqslant A_0(\varepsilon). \tag{21}$$

Now let A be any fixed arbitrary number $\geqslant \max\{1+\eta, A_0(\varepsilon)\}$. Then

$$\int_{-\infty}^{\infty} |S_\tau(\chi; x) - \chi(x)|^p \, dx < \int_{-A}^{A} |S_\tau(\chi; x) - \chi(x)|^p \, dx + \frac{\varepsilon}{2}. \tag{22}$$

Next we write

$$\int_{-A}^{A} |S_\tau(\chi; x) - \chi(x)|^p \, dx = \left(\int_{E_\eta} + \int_{[-A,A] \setminus E_\eta} \right) |S_\tau(\chi; x) - \chi(x)|^p \, dx.$$

From (17) it follows that $|S_\tau(\chi; x)| \leqslant 1 + 4/\pi^2$ for all $x \in \mathbb{R}$ and all $\tau > 0$. Consequently

$$\int_{E_\eta} |S_\tau(\chi; x) - \chi(x)|^p \, dx \leqslant \left(2 + \frac{4}{\pi^2}\right)^p 4\eta < \frac{\varepsilon}{4} \quad \text{if} \quad \eta < \frac{\varepsilon}{16} \left(\frac{\pi^2}{4 + 2\pi^2}\right)^p. \tag{23}$$

We choose an $\eta < (\varepsilon/16)(\pi^2/(4+2\pi^2))^p$ and use (19) to conclude that

$$\int_{[-A,A] \setminus E_\eta} |S_\tau(\chi; x) - \chi(x)|^p \, dx < 2\delta^p A < \frac{\varepsilon}{4} \tag{24}$$

if $\delta < (\varepsilon/8A)^{1/p}$ and $\tau > (1/\eta) T_0(\delta)$. The estimates (22)–(24) together show that (16) holds for the characteristic function of $[0, 1]$. The result easily extends to the characteristic function of any finite interval and indeed to any step function with compact support.

Given a function $f\colon \mathbb{R} \to \mathbb{R}$ belonging to $L^p(\mathbb{R})$ for some $p > 1$ and an arbitrary constant $\varepsilon_1 > 0$ we can find a step function Ω with compact support such that

$$\left(\int_{-\infty}^{\infty} |f(x) - \Omega(x)|^p \, dx \right)^{1/p} < \varepsilon_1.$$

Further, there exists $\tau_1(\varepsilon_1) > 0$ such that for all $\tau > \tau_1(\varepsilon_1)$

$$\left(\int_{-\infty}^{\infty} |S_\tau(\Omega; x) - \Omega(x)|^p \, dx \right)^{1/p} < \varepsilon_1.$$

Hence, in view of Lemma 5,

$$\left(\int_{-\infty}^{\infty} |S_\tau(f;x)-f(x)|^p\,dx\right)^{1/p}$$
$$\leqslant (K_p+1)\left(\int_{-\infty}^{\infty} |f(x)-\Omega(x)|^p\,dx\right)^{1/p}$$
$$+\left(\int_{-\infty}^{\infty} |S_\tau(\Omega;x)-\Omega(x)|^p\,dx\right)^{1/p}$$
$$<(K_p+2)\varepsilon_1$$

if $\tau>\tau_1(\varepsilon_1)$. Since ε_1 is arbitrary this proves Lemma 9 for functions which assume only real values. But this is a restriction which can obviously be dropped.

Remark. Lemma 9 seems to us to be a result of independent interest.

The next lemma plays a crucial role in our argument.

LEMMA 10. *Let $p>1$. If f is an entire function of exponential type τ belonging to $\mathfrak{F}^p$, then*

$$\left(\int_{-\infty}^{\infty} |f(x)|^p\,dx\right)^{1/p} \leqslant B_p\left(\frac{\pi}{\tau}\sum_{k=-\infty}^{\infty}\left|f\left(\frac{k\pi}{\tau}\right)\right|^p\right)^{1/p} \tag{25}$$

where B_p depends on p only.

We deduce it from the following result of Marcinkiewicz [10, Theorem 10], using an approximation method developped by B. M. Lewitan [9], N. I. Akhiezer and V. A. Marchenko (see [14, Sect. 4.10.3]), and L. Hörmander [8].

LEMMA 11. *If t_N is a trigonometric polynomial of degree at most N and $1<p<\infty$, then*

$$\int_{-\pi}^{\pi} |t_N(x)|^p\,dx \leqslant C'_p\,\frac{2\pi}{2N+1}\sum_{\nu=-N}^{N}\left|t_N\left(\frac{2\nu\pi}{2N+1}\right)\right|^p,$$

where C'_p depends on p only.

It is desirable to recall certain facts from [8]. If $\varphi(x):=((\sin\pi x)/\pi x)^2$ then for $f\in E^\tau$ and $h>0$ the function

$$f_h(x):=\sum_{\nu=-\infty}^{\infty}\varphi(hx+\nu)\,f\left(x+\frac{\nu}{h}\right) \tag{26}$$

is continuous and periodic with period $1/h$. Its Fourier coefficients

$$c_\nu(h) := h \int_{-1/2h}^{1/2h} f_h(x) e^{-2\pi i \nu h x}\, dx$$

vanish if $|\nu| > [\tau/2\pi h] + 1$, i.e., $f_h(x)$ is of the form

$$f_h(x) = \sum_{\nu=-N}^{N} a_\nu e^{2\pi i \nu h x}, \qquad \left(N := \left[\frac{\tau}{2\pi h} \right] + 1 \right).$$

Besides, $f_h(z) \to f(z)$ as $h \to 0$, the convergence being uniform on all compact subsets of the complex plane.

We shall also need the following property of f_h proved in [5, Lemma 3].

LEMMA 12. *If $f \in \mathfrak{F}^p(\delta)$, $p > 1$, then there exists a constant C such that*

$$|f_h(x)|^p < C\,|x|^{-1-p\delta} \qquad \text{for} \quad 0 < |x| < \frac{1}{2h}. \tag{27}$$

Proof of Lemma 10. Let h be of form $\tau/2\pi(N-1)$ where $N-1 \in \mathbb{N}$. Then $f_h(x/2\pi h)$ is a trigonometric polynomial of degree at most N and so by Lemma 11

$$\int_{-\pi}^{\pi} \left| f_h\left(\frac{x}{2\pi h}\right)\right|^p dx \leqslant C'_p \frac{2\pi}{2N+1} \sum_{\nu=-N}^{N} \left| f_h\left(\frac{\nu}{(2N+1)h}\right)\right|^p,$$

i.e.,

$$\int_{-1/2h}^{1/2h} |f_h(x)|^p\, dx \leqslant C'_p \frac{1}{(2N+1)h} \sum_{\nu=-N}^{N} \left| f_h\left(\frac{\nu}{(2N+1)h}\right)\right|^p.$$

Given $\varepsilon > 0$ there exists $L > 0$ such that

$$\int_{-\infty}^{\infty} |f(x)|^p\, dx < \int_{-L}^{L} |f(x)|^p\, dx + \varepsilon.$$

Since $f_h(x) \to f(x)$ uniformly on $[-L, L]$ as $h \to 0$ we can find $h_\varepsilon > 0$ with

$$\int_{-L}^{L} |f(x)|^p\, dx < \int_{-L}^{L} |f_h(x)|^p\, dx + \varepsilon \qquad \text{for} \quad 0 < h < h_\varepsilon.$$

Hence for $0 < h < \min\{h_\varepsilon, 1/2L\}$ we have

$$\begin{aligned}\int_{-\infty}^{\infty} |f(x)|^p\,dx &< \int_{-1/2h}^{1/2h} |f_h(x)|^p\,dx + 2\varepsilon\\ &\leqslant C_p' \frac{1}{(2N+1)h} \sum_{\nu=-N}^{N} \left|f_h\left(\frac{\nu}{(2N+1)h}\right)\right|^p + 2\varepsilon\\ &< C_p' \frac{\pi}{\tau} \sum_{\nu=-N}^{N} \left|f_h\left(\frac{\nu\pi}{\tau+3\pi h}\right)\right|^p + 2\varepsilon. \end{aligned} \tag{28}$$

In view of Lemma 12, there exist an integer $n_2 = n_2(\varepsilon)$ and $h_\varepsilon' > 0$ such that

$$\begin{aligned}\sum_{n_2 < |\nu| \leqslant N} \left|f_h\left(\frac{\nu\pi}{\tau+3\pi h}\right)\right|^p &< 2C\left(\frac{\tau+3\pi h}{\pi}\right)^{1+p\delta} \sum_{\nu=n_2+1}^{\infty} \nu^{-1-p\delta}\\ &< \frac{2C}{p\delta}\left(\frac{\tau+3\pi h}{\pi}\right)^{1+p\delta} n_2^{-p\delta}\\ &< \frac{2C}{p\delta}\left(\frac{\tau+3\pi h}{\pi}\right)^{1+p\delta} n_2^{-p\delta}\\ &< \frac{\varepsilon\tau}{\pi C_p'} \quad \text{if} \quad 0 < h < h_\varepsilon'. \end{aligned} \tag{29}$$

Further, since f, f_h are continuous and $\lim_{h\to 0} f_h = f$ uniformly on compact subsets it follows from

$$\begin{aligned}&\left|\sum_{|\nu| \leqslant n_2} \left\{\left|f_h\left(\frac{\nu\pi}{\tau+3\pi h}\right)\right|^p - \left|f\left(\frac{\nu\pi}{\tau}\right)\right|^p\right\}\right|\\ &\quad\leqslant \sum_{|\nu| \leqslant n_2} \left\{\left|f_h\left(\frac{\nu\pi}{\tau+3\pi h}\right)\right|^p - \left|f\left(\frac{\nu\pi}{\tau+3\pi h}\right)\right|^p\right.\\ &\qquad \left. + \left|f\left(\frac{\nu\pi}{\tau+3\pi h}\right)\right|^p - \left|f\left(\frac{\nu\pi}{\tau}\right)\right|^p\right\}\end{aligned}$$

that

$$\sum_{|\nu| \leqslant n_2} \left|f_h\left(\frac{\nu\pi}{\tau+3\pi h}\right)\right|^p < \sum_{|\nu| \leqslant n_2} \left|f\left(\frac{\nu\pi}{\tau}\right)\right|^p + \frac{2\varepsilon\tau}{\pi C_p'} \tag{30}$$

if h is sufficiently small. Inequality (25) is an obvious consequence of (28), (29), and (30).

Finally, we need

LEMMA 13. *Let* $p>1$. *If* $f\in\mathfrak{F}^p\cap\mathfrak{R}$, *then*

$$\lim_{\sigma\to\infty}\frac{\pi}{\sigma}\sum_{k=-\infty}^{\infty}\left|f\left(\frac{k\pi}{\sigma}\right)\right|^p=\int_{-\infty}^{\infty}|f(x)|^p\,dx.$$

Proof. Let

$$|f(x)|<\frac{C}{(|x|+1)^{1/p+\delta}}\qquad\text{for all}\quad x\in\mathbb{R}. \tag{31}$$

Given $\varepsilon>0$ we choose X_ε in $[(6C^p/\delta p\varepsilon)^{1/\delta p},\infty)$ large enough to have

$$\left|\int_{-X_\varepsilon}^{X_\varepsilon}|f(x)|^p\,dx-\int_{-\infty}^{\infty}|f(x)|^p\,dx\right|<\frac{\varepsilon}{3}. \tag{32}$$

If $j=j(\sigma)$ is the largest integer such that $j\pi/\sigma\leqslant X_\varepsilon$, then

$$\begin{aligned}\frac{\pi}{\sigma}\sum_{k=j+2}^{\infty}\left|f\left(\frac{k\pi}{\sigma}\right)\right|^p&<C^p\frac{\pi}{\sigma}\sum_{k=j+2}^{\infty}1\Big/\left(\frac{k\pi}{\sigma}\right)^{1+\delta p}\\&<C^p\left(\frac{\sigma}{\pi}\right)^{\delta p}\int_{j+1}^{\infty}\frac{1}{x^{1+\delta p}}\,dx\\&=\frac{C^p}{\delta p}\left(\frac{\sigma}{(j+1)\pi}\right)^{\delta p}\\&<\frac{\varepsilon}{6}\end{aligned} \tag{33}$$

since $(\sigma/(j+1)\pi)^{\delta p}<(1/X_\varepsilon)^{\delta p}<\delta p\varepsilon/6C^p$. Similarly

$$\frac{\pi}{\sigma}\sum_{k=-\infty}^{-j-2}\left|f\left(\frac{k\pi}{\sigma}\right)\right|^p<\frac{\varepsilon}{6}. \tag{34}$$

The property (31), the assumption on the size of X_ε, and the fact that $|f|\in\mathfrak{R}$ together imply

$$\left|\frac{\pi}{\sigma}\sum_{k=-j-1}^{j+1}\left|f\left(\frac{k\pi}{\sigma}\right)\right|^p-\int_{-X_\varepsilon}^{X_\varepsilon}|f(x)|^p\,dx\right|<\frac{\varepsilon}{3} \tag{35}$$

for all large σ. The desired result follows from (32)–(35).

3. Proof of Theorem 1

Let $\sigma > 0$ and consider $f_\sigma^* \equiv f - S_\sigma(f;\cdot)$. If $\tau \geqslant \sigma$ then by Lemma 8

$$L_\tau(f;\cdot) = L_\tau(f_\sigma^*;\cdot) + L_\tau(S_\sigma(f;\cdot);\cdot) = L_\tau(f_\sigma^*;\cdot) + S_\sigma(f;\cdot),$$

whence

$$f - L_\tau(f;\cdot) = f_\sigma^* + S_\sigma(f;\cdot) - L_\tau(f_\sigma^*;\cdot) - S_\sigma(f;\cdot) = f_\sigma^* - L_\tau(f_\sigma^*;\cdot).$$

By Lemma 7, f_σ^* belongs to $\mathfrak{F}^p$ and by Lemma 3, $L_\tau(f_\sigma^*;\cdot) \in E^\tau \cap \mathfrak{F}^p$. So using Lemma 10 we get

$$\|f - L_\tau(f;\cdot)\|_p \leqslant \|f_\sigma^*\|_p + B_p\left(\frac{\pi}{\tau}\sum_{k=-\infty}^{\infty}\left|f_\sigma^*\left(\frac{k\pi}{\tau}\right)\right|^p\right)^{1/p}.$$

Given $\varepsilon > 0$ we can, in view of Lemma 9, choose σ large enough to have

$$\|f_\sigma^*\|_p < \frac{\varepsilon}{2}.$$

Since f_σ^* belongs to $\mathfrak{R}$ too, we can then, by virtue of Lemma 13, find $\tau_0 \geqslant \sigma$ such that

$$\left(\frac{\pi}{\tau}\sum_{k=-\infty}^{\infty}\left|f_\sigma^*\left(\frac{k\pi}{\tau}\right)\right|^p\right)^{1/p} < \frac{\varepsilon}{2B_p} \qquad \text{for} \quad \tau \geqslant \tau_0.$$

Thus $\|f - L_\tau(f;\cdot)\|_p < \varepsilon$ for all large τ, i.e., Theorem 1 holds.

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